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955 L'ENFANT PLAZA NORTH, S.W. WASHINGTON, D.C. 20024

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SUBJECT: The Quaternion Representation of Rotations and Its Application In the Skylab Orbital Assembly - Case 610

DATE: March 31, 1971

FROM: P. H. Whipple

ABSTRACT

The quaternion method of describing a rotation is a four-parameter method that can be used to relate coordinate systems and describe the attitude of a space vehicle. The principal advantages of the quaternion method are that it does not possess the singular points inherent in Euler angle methods, and that it requires the evaluation of fewer elements than does the direction cosine approach. Disadvantages of the quaternion method are that it is unfamiliar to many users at this time and that its elements do not lend themselves to intuitive interpretation as do the elements of Euler angle methods.

Included in this memorandum are the fundamental characteristics of the quaternion and its application to rotations, the derivations and solutions of basic differential equations, attitude error evaluation, and a discussion of how these concepts are applied in the Skylab Orbital Assembly attitude control system, one of the first practical applications of the quaternion method.

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MEMORANDUM FOR FILE

I. Introduction

The basic problem of describing the orientation of one coordinate system with respect to another has usually been solved using either Euler angles or the direction cosine matrix. Another method, which uses a mathematical entity called a quaternion, has received attention in recent years and is to be used in the attitude control computations of the Skylab Apollo Telescope Mount Digital Computer (ATMDC). The purpose of this memorandum is to bring together in one place the necessary fundamentals of the quaternion representation and to illustrate how it is applied in the attitude control computations for the Skylab Orbital Assembly (OA). Although some brief comparative remarks will be made, extensive comparisons between quaternions and other representations, and lengthy discussions of their relative advantages and disadvantages can be found elsewhere (References 1, 2, and 3) and will not be included in this memorandum.

The fundamental characteristics of the quaternion representation are given in Section II. These include the definition of a quaternion, its basic algebraic properties, the representation of rotations by a quaternion, the derivation of the dynamical differential equations, solutions to these equations, and attitude error expressions. The application of these fundamentals to the attitude control function of the OA is discussed in Section III. Section IV contains a brief discussion of the Euler angle, direction cosine, and quaternion representations. The application of matrix algebra to quaternion products, and the solution to a basic quaternion differential equation are given in Appendices A and B in support of material given in Section II.

II. Fundamental Relationships

A. Quaternion Definition

The basis for the quaternion representation of a rotation is Euler's theorem which states that the general displacement of a rigid body with one point fixed is a rotation about some axis (Reference 4). The quaternion has four elements which completely identify the direction of the axis of rotation and the amount of rotation. The quaternion, $\underline{q}$, consists of a scalar element, q_4, and a vector, $\overline{q}$, and can be written as:

$$\underline{q} = q_4 + \overline{q}$$

$$= q_4 + \overline{i}q_1 + \overline{j}q_2 + \overline{k}q_3 \qquad (1)$$

$$= (q_4, \overline{q}) \qquad (2)$$

where $\overline{i}$, $\overline{j}$, and $\overline{k}$ are unit vectors along the X, Y, and Z axes respectively.* If, in addition

$$q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1$$

the quaternion is frequently referred to as a unit quaternion or versor. This property will be assumed for all quaternions that are used to identify rotations. However, it is frequently convenient to represent a vector as a quaternion with a scalar element equal to zero, in which case the unit magnitude property need not apply.

A quaternion used to identify a rotation by an angle θ about an axis $\overline{e}$ is written

$$\underline{q} = \cos\frac{\theta}{2} + \overline{e}\sin\frac{\theta}{2} \qquad (3)$$

$$= \cos\frac{\theta}{2} + (\overline{i}\cos\alpha + \overline{j}\cos\beta + \overline{k}\cos\gamma)\sin\frac{\theta}{2} \qquad (4)$$

---

*The parenthetical notation of (2) is a convenient form for concisely displaying the characteristics of a quaternion and will be used often in this memorandum. Note also the convention of denoting a quaternion as an underlined character, a vector as a character with a line over it, and a scalar as a character without a line.

where $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are the direction cosines of the axis of rotation, $\bar{e}$. A comparison of (1) and (4) identifies the quaternion elements as follows:

$$q_1 = \cos\alpha \sin\frac{\theta}{2}, \quad q_3 = \cos\gamma \sin\frac{\theta}{2}$$

$$q_2 = \cos\beta \sin\frac{\theta}{2}, \quad q_4 = \cos\frac{\theta}{2} \quad .$$

It is clear from (4) that the quaternion contains information not only about the axis of rotation and the magnitude, but also about the sense of the rotation. That is, the rotation quaternions for a rotation of θ about $\bar{e}$ and for $360 - \theta$ about $-\bar{e}$ are different, but only in the signs of all the elements. The smaller of the two rotations to the same endpoint always has a positive value for q_4. On the other hand, a rotation of θ about $\bar{e}$ and a rotation of $-\theta$ about $-\bar{e}$ are both described by the same quaternion, which is convenient, since these correpond to the same physical motions.

The transpose of a quaternion is defined by

$$\underline{q}^t = (q_4, -\bar{q}) = q_4 - \bar{i}q_1 - \bar{j}q_2 - \bar{k}q_3 \quad .$$

When used to identify a rotation, this corresponds to a rotation of an angle θ about the negative $\bar{e}$ axis.

$$\underline{q}^t = \cos\frac{\theta}{2} - \bar{e}\sin\frac{\theta}{2} \quad .$$

B. Quaternion Operations

1. Quaternion Algebra

The product of two quaternions, $\underline{p}$ and $\underline{q}$, is defined as the algebraic term-by-term product

$$\underline{p}\,\underline{q} = (p_4, \bar{p})\ (q_4, \bar{q}) \quad . \tag{5}$$

To evaluate this product, the products of $\bar{i}$, $\bar{j}$, and $\bar{k}$ are defined as

$$\bar{i}\bar{i} = \bar{k}\bar{k} = \bar{j}\bar{j} = -1 \ , \qquad \bar{j}\bar{k} = -\bar{k}\bar{j} = \bar{i}$$

$$\bar{i}\bar{j} = -\bar{j}\bar{i} = \bar{k} \ , \qquad \bar{k}\bar{i} = -\bar{i}\bar{k} = \bar{j} \quad .$$

After performing the multiplication of (5), it is seen that the quaternion product can be expressed as

$$\underline{p}\ \underline{q} = (p_4 q_4 - \bar{p}\cdot\bar{q},\ p_4\bar{q} + q_4\bar{p} + \bar{p} \times \bar{q}) \qquad (6)$$

where the vector products are the usual dot and cross products. Because of the vector cross product term, quaternion multiplication is not commutative.

The identity quaternion is defined such that the product of any other quaternion with it is identically equal to itself. From (6), it is seen that the identity quaternion must have a unit scalar component and zero vector component, that is, the identity quaternion is $(1,\bar{0})$ and

$$\underline{q}\ (1,\bar{0}) = \underline{q} \quad .$$

The inverse of $\underline{q}$, denoted by $\underline{q}^{-1}$, is defined by

$$\underline{q}\ \underline{q}^{-1} = (1,\bar{0}) \quad .$$

The derivative of a quaternion is defined by

$$\frac{d\underline{q}}{dt} = \dot{\underline{q}} = \dot{q}_4 + \bar{i}\dot{q}_1 + \bar{j}\dot{q}_2 + \bar{k}\dot{q}_3 \quad .$$

The following identities can now be quickly derived for a quaternion of any magnitude.

$$\underline{q}^{-1} = \underline{q}^{t}$$

$$\underline{q}\underline{q}^{-1} = \underline{q}^{-1}\underline{q}$$

$$(\underline{p}\ \underline{q})^{-1} = \underline{q}^{-1}\underline{p}^{-1}$$

$$(\underline{q}^{-1})^{-1} = \underline{q}$$

$$\frac{d}{dt}(\underline{q}^{-1}) = \left(\frac{d\underline{q}}{dt}\right)^{-1}$$

$$c = c(1,\bar{0}) = (c,\bar{0})$$

$c\underline{q} = (cq_4,\ c\bar{q})$ where c is a scalar.

2. Coordinate Transformation

Consider the quaternion product

$$\underline{r}' = \underline{q}^{-1}\ \underline{r}\ \underline{q} \tag{7}$$

where $\underline{q}$ is a unit quaternion relating two coordinate systems and $\underline{r}$ is the quaternion representation of a vector, i.e.,

$$\underline{r} = (o,\ \bar{r})\quad .$$

After performing the multiplication of (7), it is seen that the scalar element of $\underline{r}'$ is zero and hence $\underline{r}'$ is also a quaternion representation of a vector. The product of (7) has transformed the vector $\bar{r}$ from the original coordinate system to the second coordinate system.

This can be illustrated by the following simple example with the aid of Figure 1. The original coordinate system, X, Y, Z, has been rotated through a positive angle of ninety degrees to form a new coordinate system, X′, Y′, Z′. The axis of rotation, $\bar{e}$, is taken along the Z axis and a unit vector $\bar{r}$ is taken along the X axis.

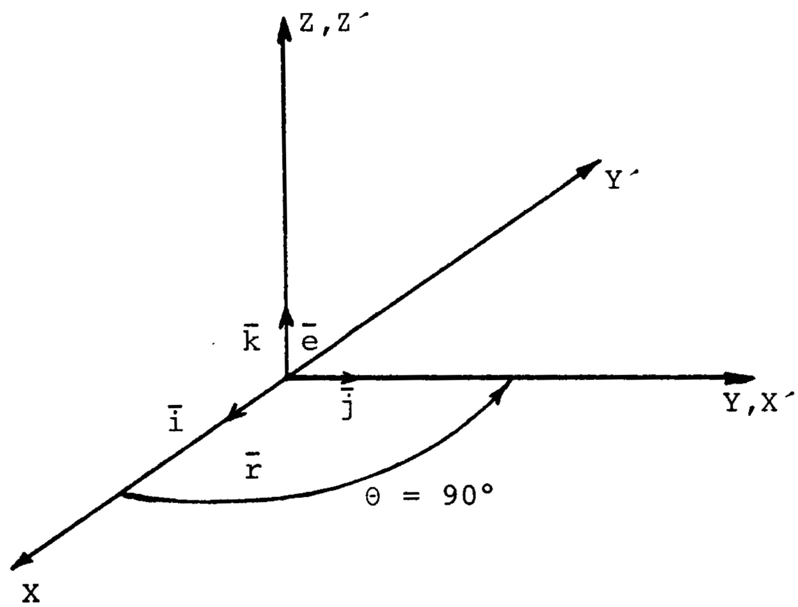

Figure 1

$$\underline{r} = (o, \bar{r}) = \bar{i}$$

$$\underline{q} = (q_4, \bar{q}) = \frac{1}{\sqrt{2}} + \frac{\bar{k}}{\sqrt{2}}$$

$$\underline{r}' = \underline{q}^{-1}\ \underline{r}\ \underline{q} = \frac{1}{2}\ (1 - \bar{k})\ (\bar{i})\ (1 + \bar{k}) = -\bar{j} \quad .$$

This indicates that the vector resolved in the second coordinate system is directed in the negative y′ direction, which is obvious from Figure 1.

The quaternion $\underline{q}$ relating two coordinate systems can be considered to be resolved in either system since the vector describing the axis of rotation is unchanged by the rotation.

The coordinate transformation of (7) can be generalized to transform a quaternion from one coordinate system to another. By substituting a quaternion, $\underline{p}$, for $\underline{r}$ in (7) and performing the multiplication, it can be seen that the scalar element of $\underline{p}$ remains unchanged and the vector component transforms as illustrated previously. This is true for a $\underline{p}$ of any magnitude.

3. Vector Rotation

It can also be shown that the product

$$\underline{r}'' = \underline{q}\ \underline{r}\ \underline{q}^{-1} \tag{8}$$

describes a rotation of a vector in a positive direction within the original coordinate system. Performing this multiplication with the same $\underline{r}$ and $\underline{q}$ as in the previous example, the following result is obtained.

$$\underline{r}'' = \bar{j} \quad .$$

The multiplication of (8) has redirected the vector from the X direction to the Y direction in the original coordinate system. This is easily verified by imagining a positive rotation of the vector of ninety degrees in Figure 1.

4. Successive Transformations

If two successive transformations between coordinate systems 1, 2, and 3 are described by

$$\underline{r}_2 = \underline{q}_{12}^{-1}\ \underline{r}_1\ \underline{q}_{12}$$

and

$$\underline{r}_3 = \underline{q}_{23}^{-1}\ \underline{r}_2\ \underline{q}_{23} \quad , \tag{9}$$

then the composite transformation from 1 to 3 can be described by

$$\underline{r}_3 = \underline{q}_{23}^{-1} (\underline{q}_{12}^{-1} \underline{r}_1 \underline{q}_{12}) \underline{q}_{23}$$

$$= (\underline{q}_{12}\underline{q}_{23})^{-1} \underline{r}_1 (\underline{q}_{12}\underline{q}_{23}) \tag{10}$$

$$= \underline{q}_{13}^{-1} \underline{r}_1 \underline{q}_{13}$$

where $\underline{q}_{13} = \underline{q}_{12}\underline{q}_{23}$. This can be extended to any number of successive coordinate transformations to give

$$\underline{q}_{1n} = \underline{q}_{12}\, \underline{q}_{23} \cdot \cdot \cdot \underline{q}_{(n-1)n} \quad . \tag{11}$$

Each of the quaternions contained in (9) and hence (10) and (11) is resolved in either of its local coordinate systems.

An alternate expression for $\underline{q}_{1n}$ can be developed in which each of the quaternions is expressed in the initial coordinate system. This can be derived by transforming each of the quaternions of (11) into the first coordinate system as discussed in Section B.2 above. The resulting expression is

$$\underline{q}_{1n} = \underline{q}_{(n-1)n}^{1} \cdot \cdot \cdot \underline{q}_{23}^{1}\, \underline{q}_{12}^{1} \tag{12}$$

where the superscript 1 indicates that the quaternion is resolved in the first coordinate system.

5. Direction Cosine Matrix

The property of (7) to perform a coordinate transformation suggests that a relationship between the direction cosine matrix and the quaternion elements can be found. This relationship can be produced by the straightforward multiplication of (7), but an easier derivation is available using the matrix representation of quaternion products as discussed in Appendix A. The quaternion representation of the direction cosine matrix, [D], is developed in Appendix A and given below.

$$[D] = \begin{bmatrix} q_1^2-q_2^2-q_3^2+q_4^2 & 2(q_1q_2+q_3q_4) & 2(q_1q_3-q_2q_4) \\ 2(q_1q_2-q_3q_4) & -q_1^2+q_2^2-q_3^2+q_4^2 & 2(q_2q_3+q_1q_4) \\ 2(q_1q_3+q_2q_4) & 2(q_2q_3-q_1q_4) & -q_1^2-q_2^2+q_3^2+q_4^2 \end{bmatrix} \quad (13)$$

The requirement for the direction cosine matrix orthonormality is given by

$$\sum_i d_{ki}d_{ji} = \delta_{kj} = \begin{cases} 1 \ , & \text{for } k=j \\ 0 \ , & \text{for } k\neq j \end{cases} \quad .$$

This equation is satisfied identically by the elements of (13) for $k\neq j$. For the cases where $k=j$, the unit magnitude constraint for the quaternion elements is necessary and sufficient.

C. Applications to Vehicle Dynamics

To maintain a knowledge of the orientation of a vehicle-fixed coordinate system relative to an inertial system, a relationship between the quaternion relating these systems and the rotational motion of the vehicle must be found. In pursuit of such a relationship, let $\bar{r}_v$ be a vehicle-fixed radius vector resolved in the vehicle system, $\bar{r}_i$ the same radius vector resolved in the inertial system, and $\underline{q}$ be the quaternion which defines the rotation of the inertial system into the vehicle system. Then

$$\underline{r}_v = \underline{q}^{-1} \ \underline{r}_i \ \underline{q}$$

and

$$\underline{r}_i = \underline{q} \ \underline{r}_v \ \underline{q}^{-1} \quad .$$

Also,

$$\underline{v}_i = \frac{d}{dt}(\underline{r}_i) = (0, \bar{v}_i)$$

$$= \dot{\underline{q}}\ \underline{r}_v\ \underline{q}^{-1} + \underline{q}\ \underline{r}_v\ \dot{\underline{q}}^{-1}$$

where $\bar{v}_i$ is the velocity vector resolved in the inertial system, and the radius vector is considered fixed in the vehicle system ($\dot{\underline{r}}_v = 0$). Continuing,

$$\underline{v}_i = \dot{\underline{q}}\ \underline{q}^{-1}\ \underline{r}_i\ \underline{q}\ \underline{q}^{-1} + \underline{q}\ \underline{q}^{-1}\ \underline{r}_i\ \underline{q}\ \dot{\underline{q}}^{-1}$$

$$= \dot{\underline{q}}\ \underline{q}^{-1}\ \underline{r}_i + \underline{r}_i\ (\dot{\underline{q}}\ \underline{q}^{-1})^{-1} \quad . \qquad (14)$$

The identity, $\underline{q}\ \underline{q}^{-1} = (1,0)$, can be differentiated to give

$$\dot{\underline{q}}\ \underline{q}^{-1} = -\underline{q}\ \dot{\underline{q}}^{-1} = -(\dot{\underline{q}}\ \underline{q}^{-1})^{-1} \quad .$$

Substituting this result into (14),

$$\underline{v}_i = \dot{\underline{q}}\ \underline{q}^{-1}\underline{r}_i - \underline{r}_i\dot{\underline{q}}\ \underline{q}^{-1}$$

$$= \underline{s}\ \underline{r}_i - \underline{r}_i\ \underline{s} \qquad (15)$$

where $\underline{s} = \dot{\underline{q}}\ \underline{q}^{-1}$ for convenience. Referring back to (6), it is seen that the only terms that survive the subtraction in (15) are the vector cross product terms. The quaternion, $\underline{s}\ \underline{r}_i - \underline{r}_i\ \underline{s}$, has the form

$$\underline{s}\ \underline{r}_i - \underline{r}_i\ \underline{s} = (0,\ 2\bar{s} \times \bar{r}_i) \quad .$$

The following vector equation can then be written.

$$\bar{v}_i = 2\bar{s} \times \bar{r}_i$$

Recalling that the velocity, radius, and angular velocity, $\dot{\theta}$, when resolved in the inertial system are related by

$$\bar{v}_i = \bar{\dot{\theta}}_i \times \bar{r}_i \quad ,$$

the following vector equation can be written.

$$\bar{\dot{\theta}}_i = 2\bar{s} \tag{16}$$

Examining $\underline{s} = (s_4,\bar{s}) = \dot{\underline{q}}\ \underline{q}^{-1}$, it is found that

$$s_4 = \dot{q}_4 q_4 + \bar{\dot{q}}\cdot\bar{q} = \sum_{i=1}^{4} \dot{q}_i q_i = \frac{1}{2} \sum_{i=1}^{4} \frac{d}{dt} q_i^2$$

$$= \frac{1}{2} \frac{d}{dt} (1) = 0 \quad .$$

Expressing (16) as a quaternion equation,

$$\underline{\dot{\theta}}_i = 2\underline{s} = 2\dot{\underline{q}}\ \underline{q}^{-1}$$

which gives

$$\dot{\underline{q}} = \frac{1}{2}\, \underline{\dot{\theta}}_i\, \underline{q} \tag{17}$$

and

$$\dot{\underline{q}} = \frac{1}{2}\,\underline{q}\,\dot{\underline{\theta}}_v \quad . \tag{18}$$

If the attitude rate sensors are mounted on the vehicle, as in a strapdown navigation scheme, expression (18) is the basic differential equation that relates the quaternion elements to the vehicle rotational motion, and is the equation that must be solved to maintain a knowledge of the vehicle attitude relative to the inertial coordinate system. If the attitude rate sensors are mounted on an inertially oriented platform, then (17) is the basic differential equation that must be used.

If an intermediate attitude reference system is used such that

$$\underline{r}_a = \underline{q}^{-1}\,\underline{r}_i\,\underline{q}$$

and

$$\underline{r}_v = \underline{p}^{-1}\,\underline{r}_a\,\underline{p} \quad ,$$

the preceding development can be used to solve for $\dot{\underline{q}}$ and a similar approach can be used to solve for $\dot{\underline{p}}$. In this case,

$$\underline{r}_i = \underline{q}\,\underline{r}_a\,\underline{q}^{-1} = \underline{q}\,\underline{p}\,\underline{r}_v\,\underline{p}^{-1}\,\underline{q}^{-1}$$

and proceeding as before,

$$\begin{aligned}
\underline{v}_i = \dot{\underline{r}}_i &= \dot{\underline{q}}\,\underline{p}\,\underline{r}_v\,\underline{p}^{-1}\,\underline{q}^{-1} + \underline{q}\,\dot{\underline{p}}\,\underline{r}_v\,\underline{p}^{-1}\,\underline{q}^{-1} \\
&\quad + \underline{q}\,\underline{p}\,\underline{r}_v\,\dot{\underline{p}}^{-1}\,\underline{q}^{-1} + \underline{q}\,\underline{p}\,\underline{r}_v\,\underline{p}^{-1}\,\dot{\underline{q}}^{-1} \\
&= \dot{\underline{q}}\,\underline{q}^{-1}\,\underline{r}_i - \underline{r}_i\,\dot{\underline{q}}\,\underline{q}^{-1} + \underline{q}\,\dot{\underline{p}}\,\underline{p}^{-1}\,\underline{q}^{-1}\,\underline{r}_i - \underline{r}_i\,\underline{q}\,\dot{\underline{p}}\,\underline{p}^{-1}\,\underline{q}^{-1} \quad .
\end{aligned}$$

In a manner analogous to the previous development for $\dot{\underline{q}}$, it can be shown that

$$\dot{\underline{\phi}}_i = 2(\dot{\underline{q}}\,\underline{q}^{-1} + \underline{q}\,\dot{\underline{p}}\,\underline{p}^{-1}\,\underline{q}^{-1})$$

where $\dot{\underline{\phi}}_i$ is the angular velocity of the vehicle coordinate system relative to the inertial system, resolved in the inertial system. Continuing as before and using the results of (17) and (18), it follows that

$$\dot{\underline{p}} = \frac{1}{2}\{\underline{p}\,\dot{\underline{\phi}}_v - \dot{\underline{\theta}}_a\,\underline{p}\} \tag{19}$$

and

$$\dot{\underline{p}} = \frac{1}{2}\{\underline{q}^{-1}\,\dot{\underline{\phi}}_i\,\underline{q} - \dot{\underline{\theta}}_a\}\,\underline{p} \tag{20}$$

where $\dot{\underline{\phi}}_v$ is the angular velocity of the vehicle coordinate system relative to the inertial system, resolved in the vehicle system, and $\dot{\underline{\theta}}_a$ is the angular velocity of the intermediate coordinate system relative to the inertial system, resolved in the intermediate system. If the attitude rate sensors are mounted on the vehicle, then (19) must be used to solve for $\underline{p}$. If the attitude rate sensors are mounted on an inertial platform, (20) must be used. Solutions to the dynamical equations (18) and (19) are given in a later section when the strapdown equations of the ATMDC are discussed.

D. Evaluation of Attitude Error

If two coordinate systems represent the desired and actual orientations of a space vehicle, then the misalignment between these systems can be described by an attitude error about each of the vehicle axes. If the quaternion $\underline{p}$ is used to identify the rotation that would rotate the desired vehicle orientation into the actual vehicle orientation, then the attitude errors can easily be determined from the elements of $\underline{p}$. If ϕ is the angle of rotation and α is the angle between the axis of rotation and the vehicle x axis, then

$$p_1 p_4 = \cos\alpha \sin\frac{\phi}{2}\cos\frac{\phi}{2} \quad . \tag{21}$$

For small angles,

$$\sin\frac{\phi}{2}\cos\frac{\phi}{2} \simeq \frac{\phi}{2} \quad .$$

Also,

$$\cos\alpha = \frac{\phi_x}{\phi}$$

where ϕ_x is the component of the rotation about the x axis, and in this case, the attitude error about the x axis. Substituting into (21), the attitude error about the x axis is then found to be

$$\phi_x = 2p_1 p_4 \quad .$$

Similarly, the attitude errors about the y and z axes are given by

$$\phi_y = 2p_2 p_4$$

and

$$\phi_z = 2p_3 p_4 \quad .$$

III. ATMDC Strapdown Calculations

A. General

The experiments that are to be performed during the Skylab mission require an accurate determination of the OA orientation and of the attitude errors. To provide this, the ATMDC accepts signals from the attitude rate sensors, which are rate gyros mounted on the OA, and repetitively executes a set of strapdown calculations that use the quaternion formulation discussed in Section II. These calculations, in conjunction with the outputs from the acquisition sun sensors, determine the attitude errors which are made available to the control system for corrective action.

Three coordinate systems are used in the implementation of the strapdown equations. These are (1) the solar inertial system, X_i, (2) the attitude reference system, X_a, and (3) the vehicle-fixed system, X_v. The solar inertial system is defined with its z axis pointed at the sun and its x axis near the orbital plane. Due to the precession of the orbit plane and the motion of the earth about the sun, the solar inertial system is not strictly an inertial system. To enable the usage of the differential equations developed in Section II which require an inertial system, the X_i system as used by the strapdown equations is modified to be a piecewise-inertial system. Each orbital sunrise, the X_i system is aligned with a true solar oriented system and maintained inertially stationary throughout the next revolution.

The vehicle-fixed coordinate system, X_v, has its z axis parallel to the solar experiment spar centerline and the x axis along the centerline of the workshop and multiple docking adapter, toward the CSM docking port.

The attitude reference coordinate system, X_a, is an intermediate system that does not necessarily have any unique physically meaningful orientation. The X_a system represents the computed desired orientation of the X_v system and any misalignment between the X_a and X_v systems indicates the presence of attitude errors. During periods when the OA is in the solar inertial attitude, the X_i and X_a systems will be parallel. During periods when the OA is in the z local vertical (Z/LV) attitude with the z axis aligned with the radius vector from the center of the earth, the X_a system will be in motion relative to the X_i system.

An attitude maneuver to a new desired attitude is accomplished by aligning the X_a and X_v coordinate systems at the beginning of the maneuver and then rotating the X_a system to a new attitude at a computed rate that will achieve the new attitude in the specified time interval. As the X_a system moves away from the X_v system toward the new attitude, attitude errors arise which, through the control system, cause the X_v system to follow the X_a system until the new attitude is achieved.

These coordinate systems are related through the quaternions $\underline{p}$ and $\underline{q}$ as follows:

$$\underline{x}_a = \underline{q}^{-1}\, \underline{x}_i\, \underline{q}$$

$$\underline{x}_v = \underline{p}^{-1}\, \underline{x}_a\, \underline{p} \quad . \tag{22}$$

In these expressions, $\underline{x}_i$, $\underline{x}_a$, and $\underline{x}_v$ are quaternion representations of a vector resolved in the X_i, X_a, and X_v coordinate systems respectively.

Near the beginning of the Skylab mission when the OA is still under control of the launch vehicle digital computer, the OA is placed in the solar inertial attitude. At this time, all three coordinate systems are parallel, and as part of the ATMDC initialization procedures, $\underline{p}$ and $\underline{q}$ are each initialized to be the identity quaternion.

$$\underline{p} = (1,0) \ , \quad \underline{q} = (1,0) \tag{23}$$

Thereafter, $\underline{p}$ and $\underline{q}$ are computed frequently to record all movements of the coordinate systems relative to each other.

B. <u>Evaluation of $\underline{q}$</u>

The quaternion $\underline{q}$ relates the attitude reference and solar inertial coordinate systems (X_a and X_i) as shown in (22), and must satisfy the differential equation derived earlier for a strapdown system as equation (18). A solution to this equation using a Taylor series expansion is given in Appendix B. The first four terms of the series, corresponding to derivatives of $\underline{q}$ through the third order, were retained in this solution. The angular velocity in this expression is the angular velocity of the X_a system, is a computed value, and is constant except for brief periods near the beginning and end of attitude maneuvers. The details of how this angular velocity is computed are given in a later section.

Assuming that the derivatives of the angular velocity are negligible, equation (B3) can be used to evaluate $\underline{q}$.

$$\underline{q}(t + \Delta t) = \underline{q}(t) + \underline{q}(t) \left\{-\frac{1}{8}(\dot{\theta})^2(\Delta t)^2 + \underline{\dot{\theta}}(\Delta t) \left(\frac{1}{2} - \frac{1}{48}(\dot{\theta})^2(\Delta t)^2\right)\right\} \quad (24)$$

In this expression, $\dot{\theta}$ is the magnitude of the angular velocity, Δt is the computation cycle time of one second, $\underline{q}(t)$ is the last previous computed value of $\underline{q}$, and $\underline{q}(t+\Delta t)$ is the new value of $\underline{q}$. In evaluating this expression in the Program Definition Document (PDD)[5], $\underline{\dot{\theta}}$ is taken as the average of the current and the last previous computed values of the angular velocity. By identifying the terms of (24) enclosed within brackets as a quaternion $\underline{\Delta q}$, $\underline{q}$ can be evaluated as indicated below:

$$\underline{q}(t + \Delta t) = \underline{q}(t) + \underline{q}(t)\underline{\Delta q} \quad (25)$$

where

$$\underline{\Delta q} = \Delta q_4 + \bar{i}\Delta q_1 + \bar{j}\Delta q_2 + \bar{k}\Delta q_3 \quad .$$

This procedure is used to evaluate $\underline{q}$ in the PDD and the expressions shown above correspond to equations 11.2.9 through 11.2.25 of the PDD. The matrix equivalent of (25) is used in the PDD to compute $\underline{q}(t+\Delta t)$.

C. Evaluation of $\underline{p}$

The quaternion $\underline{p}$ relates the attitude reference and vehicle coordinate systems (X_a and X_v) as shown in (22). Both of these systems may be in motion, relative to an inertially-fixed system, and differential equation (19) can be used to solve for $\underline{p}$. The angular velocities in this expression are the computed command rate of the X_a system, $\underline{\dot{\theta}}$, and the sensed angular velocity of the OA, $\underline{\dot{\phi}}$, obtained from the vehicle-mounted rate gyros. These angular velocities are relative to an inertial system and are resolved in the respective moving systems, X_a and X_v. Numerical integration can be used to evaluate $\underline{p}$ as shown below:

$$\underline{p}(t) = \underline{p}(t - \Delta t) + \frac{\Delta t}{2} (\underline{\dot{p}}(t) + \underline{\dot{p}}(t - \Delta t))$$

where

$$\underline{\dot{p}}(t) = \frac{1}{2} \{\underline{p}(t - \Delta t)\underline{\dot{\phi}}(t) - \underline{\dot{\theta}}(t)\underline{p}(t - \Delta t)\} \quad .$$

Normally, each of the components of $\underline{\dot{\phi}}$ is computed as the average of the outputs of two active rate gyros for the appropriate axis. The above expressions used to evaluate $\underline{p}$ correspond to equations 11.2.26 through 11.2.37 of the PDD. The computation rate for these expressions is five times per second.

To prevent the violation of the unit magnitude constraint due to accumulated round-off errors, $\underline{p}$ is normalized once per second. The normalization is done with the familiar method of dividing each element by the square root of the sum of the squares of all elements.

D. Evaluation of Attitude Errors

For small attitude errors, it was shown earlier that attitude errors about the vehicle axes could be evaluated from:

$$\phi_x = 2p_1p_4 , \qquad \phi_y = 2p_2p_4 , \qquad \phi_z = 2p_3p_4 \quad . \tag{26}$$

This expression can be simplified by recalling that

$$p_4 = \cos \frac{\phi}{2}$$

where ϕ is the total attitude error angle. For small angles, the magnitude of p_4 will be very nearly one. Theoretically, however, the sign of p_4 could be either positive or negative as can be seen from the following example. An attitude error of

$$\phi_x = -1^\circ \ , \quad \phi_y = \phi_z = 0$$

can also be described by

$$\phi_x = 359^\circ \ , \quad \phi_y = \phi_z = 0 \quad .$$

The latter convention would result in

$$P_4 = \cos \frac{\phi}{2} = \cos 179.5^\circ \simeq -1 \quad .$$

Therefore, the attitude errors may be evaluated from

$$\begin{aligned} \phi_x &= 2p_1 \text{ sign } p_4 \\ \phi_y &= 2p_2 \text{ sign } p_4 \quad . \\ \phi_z &= 2p_3 \text{ sign } p_4 \end{aligned} \tag{27}$$

These expressions are used in the PDD to compute the attitude errors, at a rate of five times a second.

An assumption made in the development of (26) and (27) is that the attitude errors remain small. Therefore, p_4 must not be allowed to vary much from its initialization value of plus one, corresponding to no attitude error, and should always remain positive for values of ϕ where (26) and (27) are valid. The attitude error expressions could then be further simplified to

$$\phi_x = 2p_1 \ , \quad \phi_y = 2p_2 \ , \quad \phi_z = 2p_3 \quad . \tag{28}$$

E. Execution of Attitude Maneuvers

Changes in attitude can be requested by the crew, ground, or ATMDC via switches or commands. The ATMDC is capable of automatically orienting the vehicle to the solar inertial or to the Z/LV attitude from any other initial attitude. Also, a maneuver to a new attitude described by three Euler angles relative to the existing attitude can be accomplished.

When an attitude maneuver is requested, the attitude reference coordinate system is aligned parallel to the vehicle system. The angular velocity of the attitude reference system is obtained by computing in sequence, the quaternion relating the desired attitude to the initial attitude, the total angle of rotation, the required magnitude of the angular velocity, and the components of the angular velocity as resolved in the vehicle system. To initially align the attitude reference system to the vehicle system, $\underline{q}$ and $\underline{p}$ are updated as follows:

$$\begin{aligned} \underline{q}' &= \underline{q}\ \underline{p} \\ \underline{q} &= \underline{q}' \\ \underline{p} &= (1,\bar{0}) \end{aligned} \tag{29}$$

Let $\underline{X}_o$ be the quaternion representation of a vector resolved in the vehicle-fixed coordinate system at the beginning of the maneuver, $\underline{X}_f$ the quaternion representation of the same vector resolved in the vehicle-fixed system in its desired orientation at the end of the maneuver, and let $\underline{c}$ be the quaternion relating these two orientations of the vehicle-fixed coordinate system. Then

$$\underline{X}_f = \underline{c}^{-1}\ \underline{X}_o\ \underline{c} \quad .$$

If the final attitude is to be the solar inertial attitude,

$$\underline{c} = \underline{q}^{-1} = \underline{q}^{t}$$

where $\underline{q}$ has been updated as in (29). If the final attitude is to be the Z/LV attitude, a further step is necessary to determine $\underline{c}$. If X_l, X_i, and X_v represent a vector resolved in the local vertical, solar inertial, and vehicle-fixed systems, respectively, then X_l can be expressed as

$$\underline{X}_l = \underline{b}^{-1} \underline{X}_i \underline{b}$$

$$= \underline{b}^{-1} (\underline{q} \underline{X}_v \underline{q}^{-1}) \underline{b} = (\underline{q}^{-1} \underline{b})^{-1} \underline{X}_v (\underline{q}^{-1} \underline{b})$$

and

$$\underline{c} = \underline{q}^{-1} \underline{b} = \underline{q}^{t} \underline{b} \quad . \tag{30}$$

The quaternion $\underline{b}$ can be computed as the product of three simpler quaternions that correspond to the Euler angles that would rotate the solar inertial attitude into the Z/LV attitude. These rotations are (1) a rotation about the vehicle Z axis to place the vehicle X axis in the orbital plane, (2) a rotation about the new vehicle X axis to place the vehicle Z axis in the orbital plane, and (3) a rotation about the Y axis to align the Z axis parallel to the radius vector. These angles are referred to as $-\nu_z$, $-\eta_x$, and $\Delta\eta_{ty}$ respectively in the PDD. From (11), the product for $\underline{b}$ is given by

$$\underline{b} = (\cos \frac{\nu_z}{2}, -\bar{k}\sin \frac{\nu_z}{2})(\cos \frac{\eta_x}{2}, -\bar{i}\sin \frac{\eta_x}{2})(\cos \frac{\Delta\eta_{ty}}{2}, \bar{j}\sin \frac{\Delta\eta_{ty}}{2}).$$

The quaternion $\underline{c}$ is then computed from (30).

If the new commanded attitude is described by three Euler angles relative to the initial attitude, $\underline{c}$ is determined in the same manner as $\underline{b}$ was determined above. The Euler angles for this maneuver are given as ξ_x, ξ_y, ξ_z about the x, y, and z axis respectively and in that order. Therefore,

$$\underline{c} = (\cos\frac{\xi_x}{2},\ \bar{i}\sin\frac{\xi_x}{2})\ (\cos\frac{\xi_y}{2},\ \bar{j}\sin\frac{\xi_y}{2})\ (\cos\frac{\xi_z}{2},\ \bar{k}\sin\frac{\xi_z}{2}).$$

Having determined the quaternion relating the final and initial attitudes, the angle of rotation, Θ, is computed from

$$\Theta = 2\tan^{-1}\left(\frac{\sqrt{1 - c_4^{\,2}}}{c_4}\right) .$$

The maneuver consists of three segments which are (1) a period of constant angular acceleration, (2) a period of constant angular velocity, and (3) a period of constant angular deceleration of the same duration as the first segment. The magnitude of the angular velocity, $\dot{\Theta}$, at any time during the maneuver is a function of Θ, Δt, the angular acceleration, and the time since the beginning of the maneuver, t. It can be shown in a straightforward manner that the duration of the angular acceleration (or deceleration) segment is given by

$$t_a = \frac{\Delta t}{2} - \sqrt{\left(\frac{\Delta t}{2}\right)^2 - \frac{\Theta}{\ddot{\Theta}}} \qquad (31)$$

where $\ddot{\Theta}$ is the angular acceleration. The magnitude of the angular velocity at any time during the maneuver is then given by

$$\dot{\Theta} = \begin{cases} \ddot{\Theta}t & , \quad 0 < t < t_a \\ \ddot{\Theta}t_a & , \quad t_a \le t \le \Delta t - t_a \\ \ddot{\Theta}(\Delta t - t) & , \quad \Delta t - t_a < t < \Delta t \end{cases} . \qquad (32)$$

Recalling the characteristics of the quaternion elements as given in Section II, it can be shown that the components of the angular velocity about each of the vehicle axes can be computed as follows.

$$\dot{\theta}_x = \frac{c_1 \dot{\theta}}{\sqrt{1 - c_4^2}}$$

$$\dot{\theta}_y = \frac{c_2 \dot{\theta}}{\sqrt{1 - c_4^2}}$$

$$\dot{\theta}_z = \frac{c_3 \dot{\theta}}{\sqrt{1 - c_4^2}}$$

These are the attitude rates at which the attitude reference system is driven during the maneuver and which are used in the evaluation of $\underline{q}$ and $\underline{p}$ discussed earlier.

A test is made early in these computations to insure that the Δt specified is sufficiently large to prevent the angular velocity from exceeding a maximum allowable value. If Δt is too small, a new Δt is computed that corresponds to an acceptable value of the constant angular velocity of the second segment of the maneuver. Also, if the final attitude is the Z/LV attitude, a fourth segment is included in the maneuver to establish the angular velocity about the vehicle Y axis necessary to maintain the Z/LV attitude. In this case, the Δt used in (31) and (32) is decreased by the duration of this fourth segment.

Other types of attitude maneuvers are necessary in the Skylab mission, such as momentum desaturation and random reacquisition maneuvers. While the details of implementing these differ somewhat from the maneuvers described above, the same concepts and techniques are applicable.

IV. Concluding Remarks

Although this memorandum is intended to be tutorial in nature and not a critical analysis of various methods of describing attitude motion, some brief comparative remarks concerning the Euler angle, quaternion, and direction cosine methods are in order.

In the Euler angle method, each of the Euler angles is evaluated by numerical integration, requiring the values of the derivatives of the Euler angles. It is the nature of every Euler angle system to possess singular points where the derivatives cannot be evaluated. These singular points are described by specific values of the second Euler angle that cause the derivatives of the first and third Euler angles to be infinite, thereby preventing evaluation of the Euler angles themselves. Therefore, there exist particular attitudes where the Euler angle system is inadequate, should the vehicle ever approach these attitudes.

The direction cosine method does not have the singularity problem of Euler angles but requires the evaluation of nine variables, or more than twice as many as in the quaternion method and three times as many as in the Euler angle method. The orthogonality conditions of the direction cosine matrix are sometimes difficult to achieve when evaluating the direction cosine elements directly and complex procedures are sometimes required to eliminate accumulated round-off errors.

The quaternion method requires the integration of one more variable than the Euler angle method but does not have the singularity problem. The quaternion method is capable of all required attitude determination and vector transformation calculations without the need for the direction cosine matrix. If, however, the direction cosine matrix were desired, it may be preferable to compute it from the quaternion elements as in (13). The orthonormality of the direction cosine matrix may be substantially easier to achieve when it is computed from the quaternion elements. As discussed in Section II, it is only necessary to normalize the quaternion. Also, computation of the direction cosine matrix from the quaternion elements is probably easier than computing these elements from Euler angles. In the latter case, the sine and cosine of each of the Euler angles is necessary. These are usually computed as truncated infinite series, requiring substantially more computations to compute the direction cosine matrix from Euler angles than from the quaternion elements.

Disadvantages of the quaternion method are that it is less familiar than the direction cosine method to most users, and that its elements do not lend themselves to intuitive interpretation as in the Euler angle method.

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Attachments

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Appendix A

Matrix Applications to Quaternion Products

A1. Matrix Representation of a Quaternion Product

The notation used in the text for a quaternion product,

$$\underline{s} = \underline{p}\ \underline{q} = (p_4 q_4 - \bar{p}\cdot\bar{q},\ p_4\bar{q} + q_4\bar{p} + \bar{p} \times \bar{q}) \ , \tag{A1}$$

is useful as a concise display of the contents of the product. However, in computing the elements of the product quaternion, this is of little value and the more convenient matrix convention is useful. The quaternion product can be expressed in either of two equivalent matrix forms.

$$\begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix} = \begin{bmatrix} p_4 & -p_3 & p_2 & p_1 \\ p_3 & p_4 & -p_1 & p_2 \\ -p_2 & p_1 & p_4 & p_3 \\ -p_1 & -p_2 & -p_3 & p_4 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} q_4 & q_3 & -q_2 & q_1 \\ -q_3 & q_4 & q_1 & q_2 \\ q_2 & -q_1 & q_4 & q_3 \\ -q_1 & -q_2 & -q_3 & q_4 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} \tag{A2}$$

For convenience, (A2) can also be expressed as

$$[s] = [M_p]\ [q] = [\hat{M}_q]\ [p]$$

where

$$[s] = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix}, \quad [M_p] = \begin{bmatrix} p_4 & -p_3 & p_2 & p_1 \\ p_3 & p_4 & -p_1 & p_2 \\ -p_2 & p_1 & p_4 & p_3 \\ -p_1 & -p_2 & -p_3 & p_4 \end{bmatrix}, \quad [\hat{M}_q] = \begin{bmatrix} q_4 & q_3 & -q_2 & q_1 \\ -q_3 & q_4 & q_1 & q_2 \\ q_2 & -q_1 & q_4 & q_3 \\ -q_1 & -q_2 & -q_3 & q_4 \end{bmatrix} .$$

As an aid in later manipulation of quaternion products, the following matrix partitioning can be done.

$$[M_p] = \left[\begin{array}{c|c} A_p & B_p \\ \hline -B_p^t & C_p \end{array}\right], \quad [\hat{M}_q] = \left[\begin{array}{c|c} A_q^t & B_q \\ \hline -B_q^t & C_q \end{array}\right]$$

where

$$A_p = \begin{bmatrix} p_4 & -p_3 & p_2 \\ p_3 & p_4 & -p_1 \\ -p_2 & p_1 & p_4 \end{bmatrix}, \quad B_p = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}, \quad C_p = [p_4] . \qquad \text{(A3)}$$

Summarizing, the quaternion product $\underline{s} = \underline{p}\,\underline{q}$ may be expressed in matrix notation by

$$[s] = [M_p]\ [q] = [\hat{M}_q]\ [p] \qquad \text{(A4)}$$

or

$$[s] = \left[\begin{array}{c|c} A_p & B_p \\ \hline -B_p^t & C_p \end{array}\right] [q] = \left[\begin{array}{c|c} A_q^t & B_q \\ \hline -B_q^t & C_q \end{array}\right] [p] \qquad \text{(A5)}$$

A2. Coordinate Transformation

It was shown in the text that the quaternion product

$$\underline{r}' = \underline{q}^{-1}\ \underline{r}\ \underline{q} \tag{A6}$$

transforms a vector, $\underline{r}$, from some initial coordinate system to a second coordinate system, where $\underline{q}$ is the quaternion relating these systems. From (A3), (A4), and (A5), this product can be expressed in matrix form as:

$$[r]' = [M_{(q^{-1})}]\ [\underline{r}\ \underline{q}] = [M_{(q^{-1})}]\ [\hat{M}_q]\ [r]$$

or

$$\begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ \hline r_4 \end{bmatrix}' = \left[\begin{array}{c|c} A_q^t & -B_q \\ \hline B_q^t & C_q \end{array}\right] \left[\begin{array}{c|c} A_q^t & B_q \\ \hline -B_q^t & C_q \end{array}\right] \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ \hline 0 \end{bmatrix}$$

$$= \left[\begin{array}{c|c} A_q^t A_q^t + B_q B_q^t & A_q^t B_q - B_q C_q \\ \hline B_q^t A_q^t - C_q B_q^t & B_q^t B_q + C_q C_q \end{array}\right] \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ \hline 0 \end{bmatrix} . \tag{A7}$$

By performing the indicated multiplications in (A7), it is seen that

$$B_q^t A_q^t - C_q B_q^t = [0\ 0\ 0]$$

$$A_q^t\ B_q - B_q\ C_q = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

and $$B_q^t\ B_q + C_q\ C_q = [1] \quad .$$

Incorporating these results into (A7) gives the two matrix equations

$$\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}' = \begin{bmatrix} A_q^t\ A_q^t + B_q\ B_q^t \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$$

$$[r_4]' = \begin{bmatrix} 0 \end{bmatrix} \quad .$$

It is seen that, as expected, the scalar element of $\underline{r}'$ is zero and that $\underline{r}'$ is indeed a vector in the new coordinate system. Also, it is clear that the direction cosine matrix for the coordinate transformation, [D], is given by

$$[D] = \begin{bmatrix} A_q^t\ A_q^t + B_q\ B_q^t \end{bmatrix}$$

$$= \begin{bmatrix} q_1^2 - q_2^2 - q_3^2 + q_4^2 & 2(q_1q_2 + q_3q_4) & 2(q_1q_3 - q_2q_4) \\ 2(q_1q_2 - q_3q_4) & -q_1^2 + q_2^2 - q_3^2 + q_4^2 & 2(q_2q_3 + q_1q_4) \\ 2(q_1q_3 + q_2q_4) & 2(q_2q_3 - q_1q_4) & -q_1^2 - q_2^2 + q_3^2 + q_4^2 \end{bmatrix} . \tag{A8}$$

In the text, it was also shown that the product

$$\underline{r}'' = \underline{q}\,\underline{r}\,\underline{q}^{-1} \qquad \text{(A9)}$$

represented the rotation of the vector $\bar{r}$ within the same coordinate system. From familiarity with the directional cosine representation of rotations, it would be expected that the rotation matrix resulting from (A9) would be the transpose of (A8). This is easily verified by substituting $-q_1$, $-q_2$, and $-q_3$ for q_1, q_2, q_3 in (A8), which is the difference between expressions (A6) and (A9).

Appendix B

Solution of A Quaternion Differential Equation

A basic differential equation that must be solved to evaluate the quaternion is derived in the text as equation (18) and is

$$\dot{\underline{q}}(t) = \frac{1}{2}\,\underline{q}(t)\underline{\dot{\theta}} \quad , \tag{B1}$$

where $\underline{\dot{\theta}}$ is the angular velocity of the moving coordinate system relative to the inertial system, as resolved in the moving system.

The Taylor series expansion for the quaternion $\underline{q}(t)$ is given by

$$\underline{q}(t + \Delta t) = \underline{q}(t) + \dot{\underline{q}}(t)\Delta t + \ddot{\underline{q}}(t)\,\frac{(\Delta t)^2}{2} + \dddot{\underline{q}}(t)\,\frac{(\Delta t)^3}{6} + \ldots \tag{B2}$$

Evaluation of the derivatives through the third order gives

$$\dot{\underline{q}}(t) = \frac{1}{2}\,\underline{q}(t)\underline{\dot{\theta}}$$

$$\ddot{\underline{q}}(t) = \frac{1}{2}\,\dot{\underline{q}}(t)\underline{\dot{\theta}} + \frac{1}{2}\,\underline{q}(t)\underline{\ddot{\theta}}$$

$$= \frac{1}{4}\,\underline{q}(t)\underline{\dot{\theta}}\;\underline{\dot{\theta}} + \frac{1}{2}\,\underline{q}(t)\underline{\ddot{\theta}}$$

$$= -\frac{(\dot{\theta})^2}{4}\,\underline{q}(t) + \frac{1}{2}\,\underline{q}(t)\underline{\ddot{\theta}}$$

$$\dddot{\underline{q}}(t) = \frac{1}{2}\,\ddot{\underline{q}}(t)\underline{\dot{\theta}} + \dot{\underline{q}}(t)\underline{\ddot{\theta}} + \frac{1}{2}\,\underline{q}(t)\underline{\dddot{\theta}}$$

$$= -\frac{(\dot{\theta})^2}{8}\,\underline{q}(t)\underline{\dot{\theta}} + \frac{1}{4}\,\underline{q}(t)\underline{\ddot{\theta}}\;\underline{\dot{\theta}} + \frac{1}{2}\,\underline{q}(t)\underline{\dot{\theta}}\;\underline{\ddot{\theta}} + \frac{1}{2}\,\underline{q}(t)\underline{\dddot{\theta}} \quad .$$

The identity

$$\underline{\dot{\theta}}\ \underline{\dot{\theta}} = (0,\dot{\bar{\theta}})\ (0,\dot{\bar{\theta}}) = (-(\dot{\theta})^2,0) = -(\dot{\theta})^2$$

was used in the above expression for the derivatives. Substituting these expressions into (B2) gives

$$\underline{q}(t + \Delta t) = q(t) + q(t)\ \{\underline{\dot{\theta}}(\frac{\Delta t}{2}) + (-\frac{(\dot{\theta})^2}{4} + \frac{\ddot{\underline{\theta}}}{2})\ \frac{(\Delta t)^2}{2}$$

$$+ (-\frac{(\dot{\theta})^2}{8}\ \underline{\dot{\theta}} + \frac{1}{4}\ \underline{\ddot{\theta}}\ \underline{\dot{\theta}} + \frac{1}{2}\ \underline{\dot{\theta}}\ \underline{\ddot{\theta}} + \frac{1}{2}\ \underline{\dddot{\theta}})\ \frac{(\Delta t)^3}{6}\}$$

If the angular velocity is constant, or very slowly changing, this expression can be simplified to

$$\underline{q}(t + \Delta t) = \underline{q}(t) + \underline{q}(t)\ \{-\frac{(\dot{\theta})^2\ (\Delta t)^2}{8} + \underline{\dot{\theta}}(\Delta t)\ (\frac{1}{2} - \frac{(\dot{\theta})^2(\Delta t)^2}{48})\}\quad . \qquad \text{(B3)}$$

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